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Puzzle Type Examples of Linear Congruence

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ABSTRACT

In this general note we solve two puzzle type examples in a systematic way to get general solution by using linear congruence and Diophantine equations.

Keywords: Diophantine equation; linear congruence; incongruent solutions; puzzle.

1. INTRODUCTION

Number Theory sometimes called Higher Arithmetic is the oldest and natural branch of Mathematics. Though it is the purest branch of mathematics, it has applications in cryptography, computer science, algebraic geometry, in-formation theory and real world problems. Number theory can be understood by laypersons and amateurs can also enjoy it, which is the beauty of this subject. Great German Mathematician Karl Friedrich Gauss (1777-1855) called it the Queen of Mathematics[1,4]. A congruence which is refined statement of divisibility plays crucial role in number theory. This concept of congruence was also first introduced by Gauss[2]. The study of linear congruences is similar to the study of linear Diophantine equations in two unknowns. In this article we discuss linear congruence and somewhat interesting puzzle type examples on it. We solve these puzzle type examples in a systematic way rather to solve by hit and mess method. For other interesting examples readers are referred to[3].

2. PRELIMINARIES

2.1 The Diophantine Equation

Any equation in one or more unknowns that is to be solved in integers is called Diophantine equation. The linear Diophantine equation in two unknowns is[1,4].

$$ax + by = c \tag{2.1}$$

where a, b, c are integers and a, b not both zero.

Given Diophantine equation can have a number of solutions, as is the case with $2x + 4y = 12$, where

$2(4) + 4(1) = 12,$
$2(-6) + 4(6) = 12,$
$2(10) + 4(-2) = 12.$

On the other hand $2x + 4y = 3$ has no solution. The following points pertaining to Diophantine equation (2.1) are to be remembered:

- The linear Diophantine equation (2.1) has a solution if and only if $d \mid c$ where $d = \gcd(a, b)$ i. e. the greatest common divisor of a and b.
- If (x_0, y_0) is one solution of (2.1) and $d = \gcd(a, b)$ then $x_1 = x_0 + \frac{b}{d}t$ and $y_1 = y_0 - \frac{a}{d}t$ is the general solution of equation (2.1).

2.2 Linear Congruence

An expression of the form[1,2,4]

$$ax \equiv b(mod\ n) \tag{2.2}$$

is called a linear congruence $mod\ n$.

An integer x_0 for which $ax_0 \equiv b(mod\ n)$ is called a solution of (2.2).
Now as, $ax_0 \equiv b(mod\ n)$

$\Rightarrow n \mid (ax_0 - b)$

$\Rightarrow \exists$ an integer y_0 such that $ax_0 - b = ny_0$

$\Rightarrow ax_0 - ny_0 = b.$

Thus, (x_0, y_0) is a solution of the Diophantine equation $ax - ny = b$.

Therefore, the linear congruence (2.2) is equivalent to the linear Diophantine equation $ax - ny = b$. We can then observe that, if x_0 is a particular solution of (2.2), then $x = x_0 + \frac{n}{d}t$, t being an integer is its general solution.

- If $d = \gcd(a, n)$ and $d \mid b$ then the linear congruence (2.2) has exactly d incongruent solutions $(mod\ n)$.
- Incongruent solutions are the distinct solutions $(modulo\ n)$.

3. EXAMPLES

3.1 Puzzle of Breads

A farmer's wife makes certain number of breads for her three sons. She leaves home along with her husband for outside work. The three sons were engaged at three different places in their routine work. When the eldest among them returns, as per their convention, offers one bread to their pet dog. He then equally distributes the remaining breads into three parts without cutting any bread into pieces. i.e. each part contains integral number of breads. He eats one portion, keeps back remaining breads for his brothers and leaves. The second son returns home, thinking that he is the first to come, offers one bread to the dog, divides the remaining breads into three equal parts (each part with integral number of breads), eats one portion, keeps back remaining and leaves. The youngest of them follows the same practice. In the evening, when they all come together, offer one bread to their dog, distribute the remaining breads into three equal parts and share among themselves. If no bread was cut into pieces while distributing, then find the number of breads that were made by the farmer's wife.

To solve this, it is assumed that the number of breads is in double figures (which seems practical). Let x be the number of breads made by farmer's wife. At first stage, the number of remaining breads is $\frac{2x-2}{3}$. At second stage, the number of remaining breads is $\frac{4x-10}{9}$. At third stage, the number of remaining breads is $\frac{8x-38}{27}$. Lastly after offering one bread to dog, the number $\frac{8x-38}{27} - 1 = \frac{8x-65}{27}$ of remaining breads is divisible by 3.

$\therefore \frac{8x-65}{27} = 3k,$ where k is positive integer.

$$\Rightarrow 8x - 81k = 65 \tag{3.1}$$

This is linear Diophantine equation. It is bit difficult to solve by inspection. We solve it as follows:

By (3.1), $x = \frac{81k+65}{8}$.

So to find x, we need to solve the linear congruence

$81k \equiv -65 (mod\ 8)$.

Again by (3.1), $8(x - 11k - 8) + 7k = 1$

$$\Rightarrow 8u + 7k = 1 \tag{3.2}$$

where $u = x - 11k - 8$ (a)

Using (3.2) we get, $7(u + k - 1) + u = -6$

$$\Rightarrow 7v + u = -6 \tag{3.3}$$

where $v = u + k - 1$ (b)

The linear Diophantine equation (3.3) has many solutions and can be solved easily by inspection. e.g. $v = 0$ and $u = -6$ is one solution. Keeping in mind that the linear congruence $81k \equiv -65(mod\, 8)$ has exactly one incongruent solution and k is positive integer, equation (b) with the values of v and u gives us the value of k as $7(mod\, 8)$.

Therefore the general solution of $81k \equiv -65(mod\, 8)$ is given by $7 + \frac{n}{d}t$ where $n = 8$ and $d = \gcd(a, n) = \gcd(81, 8) = 1$ and $t = 0, 1, 2, ...$ Clearly all the solutions of $81k \equiv -65(mod\, 8)$ are congruent.

Thus, $7 + 8t$ is the general solution of $81k \equiv -65(mod\, 8)$.

$\therefore \; x = \frac{81(7+8t)+65}{8}$ where $t = 0, 1, 2, ...$

i.e. $x = 79, 160, 241, ...$

$\therefore$ 79 breads were made by farmers wife.

(Other answers are practically impossible!) □

3.2 Puzzle of Beads

Once a merchant buys specific, beautiful beads in double figures for his three dear wives. He keeps them in his trousers pocket, hangs it near the bed and goes to sleep. Late in the night, when his first wife finds him asleep, checks the pocket to find beautiful beads. She divides them into three equal parts and keeps one for herself. The second wife, after a while, checks the pocket, finds the beads, divides them into three equal parts and keeps one for herself. The third wife follows the same practice. The merchant wakes up in the morning and verifies the beads in his pocket. It astonished him to find them least than he actually bought. To surprise his wives, he divides the remaining beads in three equal parts with one bead in excess. The quiz is how many beads were bought by the merchant?

Letting x as the number of beads bought by merchant. It can be seen that

$x = \frac{81k+27}{8}$ where k is positive integer.

The actual problem is to solve the linear congruence $81k \equiv -27(mod\ 8)$. It can be solved on similar line as the solution of first example. After solving this linear congruence, we eventually get

$x = \frac{81(5+8t)+27}{8}$ where $t = 0, 1, 2, ...$

i. e. $x = 54, 135, 216, ...$
The required number of beads bought by merchant is 54. □

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