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Study of Existence and uniqueness of solution of abstract nonlinear differential equation of finite delay

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Abstract

In this paper, we study the existence and uniqueness of solution of differential equation of finite delay with nonlocal condition in cone metric space . The result is obtained by using the some extensions of Banach's contraction principle in complete cone metric space.

1 Introduction

The purpose of this paper is study the existence and uniqueness of solution of inhomogeneous semilinear evolution equation with nonlocal condition in cone metric space of the form:

$$x'(t) = Ax(t) + f\big(t, x(t), x(t-1)\big), \quad t \in J = [0, b] \tag{1.1}$$

$$x(t-1) = \psi(t) \quad 0 \le t < 1. \tag{1.2}$$

$$x(0) + g(x) = x_0, \tag{1.3}$$

where A is an infintesimal generator of strongly continuous semigroup of bounded linear operator T(t) in X with domain $D(A)$, the unknown $x(\cdot)$ takes values in the Banach space X; $f : J \times X \times X \to X$, $g : C(J, X) \to X$ are appropriate continuous functions and x_0 is given element of X. $\psi(t)$ is a continuous function for $0 \le t < 1$, $\lim_{t \to 1-0} \psi(t)$ exists, for which we denote by $\psi(1-0) = c_0$. if we observed a function $x(t-1)$ which is unable to define as solution for $0 \le t < 1$. Hence, we have to impose some condition, for example the condition (1.2). We note that, if $0 \le t < 1$, the problem is reduced to integrodifferential equation

$$x'(t) = Ax(t) + f(t, x(t), \psi(t))$$

with initial condition $x(0) + g(x) = x_0$. Here, it is essential to obtain the solutions of (1.1)–(1.3) for $0 \le t < b$.

The objective of the present paper is to study the existence and uniqueness of solution of the evolution equation (1.1)–(1.3) under the conditions in respect of the cone metric space and fixed point theory. Hence we extend and improve some results reported in [6].

The paper is organized as follows: we discuss the preliminaries. we dealt with study of existence and uniqueness of solution of inhomogeneous evolution equation with nonlocal condition in cone metric space.

2 Preliminaries

Let us recall the concepts of the cone metric space and we refer the reader to [1, 2, 3, 4, 5, 6] for the more details.

Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if,

1. P is closed, nonempty and $P \neq \{0\}$;
2. $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
3. $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we define a partial ordering relation $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for every $x, y \in E$. The least positive number satisfying above is called the normal constant of P.

In the following we always suppose E is a real Banach space , P is a cone in E with $intP \neq \phi$, and $\leq$ is partial ordering with respect to P.

Definition 2.1 Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of cone metric space is more general than that of metric space.

The following example is a cone metric space, see [**?**].

Example 2.2 Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.3 Let X be a an ordered space. A function $\Phi : X \to X$ is said to a comparison function if for every $x, y \in X$, $x \leq y$, implies that $\Phi(x) \leq \Phi(y)$, $\Phi(x) \leq x$ and $\lim_{n\to\infty} \|\Phi^n(x)\| = 0$, for every $x \in X$.

Example 2.4 Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$. It is easy to check that $\Phi : E \to E$, with $\Phi(x, y) = (ax, ay)$, for some $a \in (0, 1)$ is a comparison function. Also if Φ_1, Φ_2 are two comparison functions over $\mathbb{R}$, then $\Phi(x, y) = (\Phi_1(x), \Phi_2(y))$ is also a comparison function over E.

3 Existence and uniqueness of solution

Let X is a Banach space with norm $\|\cdot\|$. Let $B = C(J, X)$ be the Banach space of all continuous functions from J into X endowed with supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in J\}.$$

Let $P = \{(x, y) : x, y \geq 0\} \subset E = \mathbb{R}^2$ be a cone and define $d(f, g) = (\|f - g\|_\infty, \alpha\|f - g\|_\infty)$, for every $f, g \in B$. Then it is easily seen that (B, d) is a cone metric space.

Definition 3.1 *The function $x \in B$ satisfies the integral equation*
case I *:for $0 \leq t < 1$*

$$x(t) = T(t)[x_0 - g(x)] + \int_0^1 T(t - s)f(s, x(s), x(s - 1))ds, \tag{3.1}$$

case II *:for $1 \leq t < b$*

$$\begin{aligned} x(t) = T(t)[x_0 - g(x)] + \int_0^1 T(t - s)f(s, x(s), x(s - 1))ds \\ +T(t)[x_0 - g(x)] + \int_1^t T(t - s)f(s, x(s), x(s - 1))ds, \end{aligned} \tag{3.2}$$

is called the mild solution of the evolution equation (1.1)–(1.3).

We need the following lemma for further discussion:

Lemma 3.2 *[5] Let (X, d) be a complete cone metric space, where P is a normal cone with normal constant K. Let $f : X \to X$ be a function such that there exists a comparison function $\Phi : P \to P$ such that*

$$d(f(x), f(y)) \leq \Phi(d(x, y)),$$

for every $x, y \in X$. Then f has a unique fixed point.

We list the following hypotheses for our convenience:

(H_1) A is an infintesimal generator of strongly continuous semigroup of bounded linear operator T(t) in X for each $t \in J$, and hence there exists a constant K such that

$$K = \sup_{t\in J} \|T(t)\|.$$

(H_2) Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a comparison function.

(i) There exists continuous function $p_1, p_2 : J \to \mathbb{R}^+$ such that
case I :for $0 \leq t < 1$

$$\Big(\|f(t,x(t),\psi(t)) - f(t,y(t),\psi(t))\|, \alpha\|f(t,x(t),\psi(t)) - f(t,y(t),\psi(t))\|\Big)$$

$$\leq p_1(t)\Phi\Big(d(x,y)\Big),$$

case II :for $1 \leq t < b$

$$\Big(\|f(t,x(t),x(t-1)) - f(t,y(t),y(t-1))\|, \alpha\|f(t,x(t),x(t-1)) - f(t,y(t),y(t-1))\|\Big)$$

$$\leq p_2(t)\Phi\Big(d(x,y)\Big),$$

$$\Big(\|g(x) - g(y)\|, \alpha\|g(x) - g(y)\|\Big) \leq G\Phi\Big(d(x,y)\Big),$$

for every $t \in J$ and $x, y \in X$.

(H_3) $\sup_{t\in J}\Big[KG + \int_0^t K[p_1(s) + p_2(s)]ds\Big] = 1.$

Theorem 3.3 *Assume that hypotheses $(H_1) - (H_3)$ hold. Then the evolution equation (1.1)–(1.2) has a unique solution x on J.*

Proof: The operator $F : B \to B$ is defined by
case I :for $0 \leq t < 1$

$$Fx(t) = T(t)[x_0 - g(x)] + \int_0^1 T(t-s)f(s,x(s),x(s-1))ds, \tag{3.3}$$

case II :for $1 \leq t < b$

$$Fx(t) = T(t)[x_0 - g(x)] + \int_0^1 T(t-s)f(s,x(s),x(s-1))ds$$

$$+T(t)[x_0 - g(x)] + \int_1^t T(t-s)f(s,x(s),x(s-1))ds, \tag{3.4}$$

By using the hypotheses $(H_1) - (H_3)$, we have
case I :for $0 \leq t < 1$

$$\begin{aligned}
&\Big(\|Fx(t) - Fy(t)\|, \alpha\|Fx(t) - Fy(t)\|\Big) \\
&\leq \Big(\|T(t)\|\|g(x) - g(y)\| + \int_0^1 \|T(t-s)\|\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|ds, \\
&\quad \alpha\|T(t)\|\|g(x) - g(y)\| + \alpha\int_0^1 \|T(t-s)\|\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|ds\Big) \\
&\leq \|T(t)\|\Big(\|g(x) - g(y)\|, \alpha\|g(x) - g(y)\|\Big) \\
&\quad + \int_0^t K\Big(\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|, \alpha\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|\Big)ds \\
&\leq KG\Phi\Big(\|x-y\|, \alpha\|x-y\|\Big) + \int_0^t Kp_1(s)\Phi\Big(\|x(s)-y(s)\|, \alpha\|x(s)-y(s)\|\Big)ds \\
&\leq KG\Phi\Big(\|x-y\|_\infty, \alpha\|x-y\|_\infty\Big) + \Phi\Big(\|x-y\|_\infty, \alpha\|x-y\|_\infty\Big)\int_0^t Kp_1(s)ds \\
&\leq KG\Phi\Big(d(x,y)\Big) + \Phi\Big(d(x,y)\Big)\int_0^t Kp_1(s)ds \\
&\leq \Phi\Big(d(x,y)\Big)\Big[KG + \int_0^t Kp_1(s)ds\Big] \\
&\leq \Phi\Big(d(x,y)\Big)\Big[KG + \int_0^t K[p_1(s) + p_2(s)]ds\Big] \\
&\leq \Phi\Big(d(x,y)\Big),
\end{aligned} \tag{3.5}$$

By using the hypotheses $(H_1) - (H_3)$, we have
case II :for $1 \leq t < b$

$$\begin{aligned}
&\Big(\|Fx(t) - Fy(t)\|, \alpha\|Fx(t) - Fy(t)\|\Big) \\
&\leq \Big(\|T(t)\|\|g(x) - g(y)\| + \|T(t-s)\| \\
&\quad \times \Big[\int_0^1 \|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|ds + \int_1^t \|f(s,x(s),x(s-1)) - f(s,y(s),y(s-1))\|ds\Big], \\
&\quad \alpha\|T(t)\|\|g(x) - g(y)\| + \alpha\|T(t-s)\| \\
&\quad \times \Big[\int_0^1 \|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|ds + \int_1^t \|f(s,x(s),x(s-1)) - f(s,y(s),y(s-1))\|ds\Big]\Big) \\
&\leq \|T(t)\|\Big(\|g(x) - g(y)\|, \alpha\|g(x) - g(y)\|\Big) \\
&\quad + \int_0^1 K\Big(\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|, \alpha\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|\Big)ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_1^t K\Big(\|f(s,x(s),x(s-1))-f(s,y(s),y(s-1))\|,\alpha\|f(s,x(s),x(s-1))-f(s,y(s),y(s-1))\|\Big)ds\\
&\le KG\Phi\Big(\|x-y\|_\infty,\alpha\|x-y\|_\infty\Big)+\Phi\Big(\|x-y\|_\infty,\alpha\|x-y\|_\infty\Big)[\int_0^1 Kp_1(s)ds+\int_1^t Kp_2(s)ds]\\
&\le KG\Phi\Big(d(x,y)\Big)+\Phi\Big(d(x,y)\Big)[\int_0^1 K(p_1(s)+p_2(s))ds+\int_1^t K(p_1(s)+p_2(s))ds]\\
&\le \Phi\Big(d(x,y)\Big)\Big[KG+\int_0^t K[p_1(s)+p_2(s)]ds\Big]\\
&\le \Phi\Big(d(x,y)\Big) \qquad (3.6)
\end{aligned}
$$

for every $x, y \in B$. This implies that $d\Big(Fx, Fy\Big) \le \Phi\Big(d(x,y)\Big)$, for every $x,\ y \in B$. Now an application of Lemma 3.2, the operator has a unique point in B. This means that the equation (1.1)–(1.2) has unique solution. This completes the proof of the Theorem 3.3.

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